

ON A CONJECTURE OF TSFASMAN AND AN INEQUALITY OF SERRE FOR THE NUMBER OF POINTS OF HYPERSURFACES OVER FINITE FIELDS

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Dedicated to Misha Tsfasman and Serge Vlăduț on their 60th birthdays

ABSTRACT. We give a short proof of an inequality, conjectured by Tsfasman and proved by Serre, for the maximum number of points of hypersurfaces over finite fields. Further, we consider a conjectural extension, due to Tsfasman and Boguslavsky, of this inequality to an explicit formula for the maximum number of common solutions of a system of linearly independent multivariate homogeneous polynomials of the same degree with coefficients in a finite field. This conjecture is shown to be false, in general, but is also shown to hold in the affirmative in a special case. Applications to generalized hamming weights of projective Reed-Muller codes are outlined and a comparison with an older conjecture of Lachaud and a recent result of Couvreur is given.

1. INTRODUCTION

What is the maximum number of \mathbb{F}_q -rational points that a hypersurface of degree d in m -space over the finite field \mathbb{F}_q with q elements can have? An intuitive approach could be to project the m -space onto an $(m-1)$ -space. If this projection map from the hypersurface to the $(m-1)$ -space is flat, then a point below has at most d points above on the hypersurface. This suggests that d times the number of \mathbb{F}_q -rational points in the $(m-1)$ -space is a natural upper bound. More precisely, if $f \in \mathbb{F}_q[x_1, \dots, x_m]$ is of degree d and $Z(f) := \{P \in \mathbb{A}^m(\mathbb{F}_q) : f(P) = 0\}$ the corresponding affine hypersurface or if $F \in \mathbb{F}_q[x_0, x_1, \dots, x_m]$ is a (nonzero) homogeneous polynomial of degree d and $V(F) := \{P \in \mathbb{P}^m(\mathbb{F}_q) : F(P) = 0\}$ the corresponding projective hypersurface, then

$$(1) \quad |Z(f)| \leq dq^{m-1} \quad \text{and} \quad |V(F)| \leq dp_{m-1},$$

where for any $j \in \mathbb{Z}$, we have set

$$(2) \quad p_j := |\mathbb{P}^j(\mathbb{F}_q)| = q^j + q^{j-1} + \dots + q + 1 \text{ if } j \geq 0 \text{ and } p_j := 0 \text{ if } j < 0.$$

The bounds in (1) are true and a precise proof can be easily given using double induction on d and m ; see, e.g., [12, pp. 275–276]. Variants of the bound in (1) for $|Z(f)|$ are also known in the literature as Schwarz-Zippel Lemma or more elaborately, as Schwarz-Zippel-DeMillo-Lipton Lemma, although it can be traced more than 50 years earlier to the work of O. Ore in 1922; see, e.g., [12, p. 320]. If $d > q$, the bound dq^{m-1} exceeds $|\mathbb{A}^m(\mathbb{F}_q)|$ and is hence uninteresting, whereas if $d \leq q$, then it is attained as can be seen readily by considering the polynomial $g_d(x_1, \dots, x_m) = (x_1 - a_1) \cdots (x_1 - a_d)$, where a_1, \dots, a_d are distinct elements of \mathbb{F}_q .

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Thus dq^{m-1} is the maximum value for $|Z(f)|$ when $d \leq q$. Likewise in the projective case, if $d \geq q+1$, the bound dp_{m-1} exceeds $|\mathbb{P}^m(\mathbb{F}_q)|$ and so it is uninteresting in that case. However, if $d \leq q$ and a_1, \dots, a_d are as before, then it is easy to see that the analogous homogeneous polynomial $G_d(x_0, x_1, \dots, x_m) = (x_1 - a_1 x_0) \cdots (x_d - a_d x_0)$ has exactly $dq^{m-1} + p_{m-2}$ zeros in $\mathbb{P}^m(\mathbb{F}_q)$. Also if $d = q+1$, then $G_{q+1} := x_0 G_q$ has exactly $dq^{m-1} + p_{m-2}$ zeros in $\mathbb{P}^m(\mathbb{F}_q)$. But $dq^{m-1} + p_{m-2} < dp_{m-1}$ if $d > 1$ and $m > 1$. Moreover, an example of a homogeneous polynomial of degree $d \leq q$ with exactly dp_{m-1} zeros is difficult to come by. Motivated perhaps by this, M. A. Tsfasman made a conjecture, in the late 80's, that $dq^{m-1} + p_{m-2}$ is the “true upper bound”. In other words,

$$(3) \quad |V(F)| \leq dq^{m-1} + p_{m-2} \quad \text{and hence} \quad \max_F |V(F)| = dq^{m-1} + p_{m-2} \quad \text{if } d \leq q+1,$$

where F varies over homogeneous polynomials of degree d in $\mathbb{F}_q[x_0, x_1, \dots, x_m]$. This conjecture was soon proved in the affirmative by J.-P. Serre [13], and thus we will refer to the inequality in (3) as Serre's inequality. An alternative proof of (3) was also given later by Sørensen [14]. Some years later, Tsfasman and Boguslavsky [2] formulated more general conjectures for the maximum number of points in $\mathbb{P}^m(\mathbb{F}_q)$ of systems of polynomial equations of the form

$$(4) \quad \begin{cases} F_1(x_0, x_1, \dots, x_m) = 0 \\ \vdots \\ F_r(x_0, x_1, \dots, x_m) = 0 \end{cases}$$

A quantitative version of their conjecture essentially states that if $d < q-1$, then

$$(5) \quad \max_{F_1, \dots, F_r} |V(F_1, \dots, F_r)| = p_{m-2j} + \sum_{i=j}^m \nu_i (p_{m-i} - p_{m-i-j}),$$

where F_1, \dots, F_r vary over linearly independent homogeneous polynomials of degree d in $\mathbb{F}_q[x_0, x_1, \dots, x_m]$, and where $(\nu_1, \dots, \nu_{m+1})$ is the r th element of the set

$$\Lambda(d, m) := \{(\alpha_1, \dots, \alpha_{m+1}) \in \mathbb{Z}^{m+1} : \alpha_1, \dots, \alpha_{m+1} \geq 0 \text{ and } \alpha_1 + \dots + \alpha_{m+1} = d\}$$

ordered in descending lexicographic order, and where $j := \min\{i : \nu_i \neq 0\}$.

Observe that $(d, 0, \dots, 0)$ is lexicographically the first element of $\Lambda(d, m)$ and so (5) reduces to (3) in the case $r = 1$, and is thus true, thanks to Serre [13] and Sørensen [14]. The conjecture was proved in the affirmative in the next case of $r = 2$ by Boguslavsky in 1997. Also (5) holds trivially when $d = 1$. But the general case appears to have been open for almost two decades. Moreover, some auxiliary conjectures given in [2] (see also Remark 3.6 in Section 3 below) that would imply the Tsfasman-Boguslavsky Conjecture have also been open.

We are now ready to describe the main results of this paper. In Section 2 below, we give a very short proof of Serre's inequality. Serre's original proof is quite elegant and Sørensen's proof has its merits and applications. Both the proofs involve some clever double counting argument, which are avoided in the short proof given here, thus making it comparable to the elementary proofs of (1) that one can find in textbooks [12, Theorems 6.13, 6.15]. Next, we consider the Tsfasman-Boguslavsky Conjecture (TBC) stated above and show that while it is true if $d = 2$ and $r \leq m+1$, it is false, in general, when $d = 2$ and $r \geq m+2$. In a forthcoming paper [4] it is shown that the TBC is true for any positive integer d , provided $r \leq m+1$. In the last section, we outline connections with coding theory and show that the TBC and in particular, our results are intimately related to the explicit determination of generalized hamming weights of projective Reed-Muller codes. We also compare the TBC with a recent result of Couvreur [3] on an upper bound for the number of \mathbb{F}_q -rational points of arbitrary projective varieties defined over \mathbb{F}_q .

A key ingredient for us is the work of Zanella [17] on the number of \mathbb{F}_q -rational points on intersections of quadrics or equivalently, on linear sections of the variety defined by the quadratic Veronese embedding $\mathbb{P}^m \hookrightarrow \mathbb{P}^M$, where $M = \binom{m+2}{2} - 1$.

2. SERRE'S INEQUALITY

Throughout this paper, m denotes a positive integer, q a prime power, \mathbb{F}_q the field with q elements. Also for any nonnegative integer j , we will denote by \mathbb{P}^j the j -dimensional projective space over \mathbb{F}_q , and by $\widehat{\mathbb{P}}^j$ its dual, consisting of all hyperplanes in \mathbb{P}^j . Note that $|\mathbb{P}^j(\mathbb{F}_q)| = |\widehat{\mathbb{P}}^j(\mathbb{F}_q)| = p_j$, where p_j is as in (2). The following lemma is due to Zanella [17, Lemma 3.3]. A proof is included for the sake of completeness. Alternative proofs are indicated in Remark 2.3 below.

Lemma 2.1. *Let $X \subseteq \mathbb{P}^m$ and $a := \max\{|X \cap H| : H \in \widehat{\mathbb{P}}^m\}$. Then $|X| \leq aq + 1$.*

Proof. Induct on m . The case $m = 1$ being trivial, assume that $m > 1$ and that the result holds for smaller values of m . Let $H^* \in \widehat{\mathbb{P}}^m$ be such that $|H^* \cap X| = a$. Let $Y := H^* \cap X$ and let H' be a hyperplane of H^* such that $b := |H' \cap Y|$ is the maximum among the cardinalities of all hyperplane sections of Y in $H^* \simeq \mathbb{P}^{m-1}$. By induction hypothesis, $a \leq bq + 1$. Now we write $|X| = |H' \cap X| + |X \setminus (H' \cap X)|$ and observe that on the one hand $|H' \cap X| = |H' \cap Y| \leq b$, whereas on the other hand, every $P \in X \setminus (H' \cap X)$ is contained in a unique $H \in \widehat{\mathbb{P}}^m$ with $H \supset H'$. Thus

$$|X \setminus (H' \cap X)| \leq \left| \bigcup_{H \supset H'} (H \cap X) \setminus (H' \cap X) \right| \leq (q+1)(a-b),$$

where the last inequality follows by noting that the number of $H \in \widehat{\mathbb{P}}^m$ containing a fixed codimension 2 linear subspace H' of \mathbb{P}^m is $q+1$ and also noting that $|H \cap X| - |H' \cap X| \leq a - |H' \cap Y| = a - b$ for any such H . Since $a \leq bq + 1$, it follows that $|X| \leq b + (q+1)(a-b) = aq + a - bq \leq aq + 1$, as desired. \square

Theorem 2.2 (Serre). *If $F \in \mathbb{F}_q[x_0, x_1, \dots, x_m]$ is homogeneous of degree d , then $|V(F)| \leq dq^{m-1} + p_{m-2}$. Consequently, $\max_F |V(F)| = dq^{m-1} + p_{m-2}$ if $d \leq q+1$.*

Proof. We use induction on m . The case $m = 1$ is trivial. Thus we assume $m > 1$ and consider the following two cases.

Case 1: $V(F)$ does not contain any hyperplane in \mathbb{P}^m .

In this case $F|_H \neq 0$ for all $H \in \widehat{\mathbb{P}}^m$. So the validity of the result with m replaced by $m-1$ implies $|V(F) \cap H| \leq dq^{m-2} + p_{m-3}$ for all $H \in \widehat{\mathbb{P}}^m$. Hence by Lemma 2.1, $|V(F)| \leq dq^{m-1} + p_{m-2}$. Thus the desired inequality follows by induction on m .

Case 2: $V(F)$ contains a hyperplane in \mathbb{P}^m , say $H = V(h)$.

In this case $|V(F) \cap H| = |H| = p_{m-1}$. We will now estimate $|V(F) \cap H^c|$. First, by a suitable linear change of coordinates in \mathbb{P}^m assume that $h = x_0$. Since $F|_H = 0$, we can write $F = x_0 G$ for some homogeneous $G \in \mathbb{F}_q[x_0, \dots, x_m]$ of degree $d-1$. Moreover $V(F) \cap H^c$ corresponds to the zeros in $\mathbb{A}^m(\mathbb{F}_q)$ of the polynomial $G(1, x_1, \dots, x_m)$. Hence $|V(F) \cap H^c| \leq (d-1)q^{m-1}$, thanks to (1). Consequently,

$$|V(F)| = |V(F) \cap H^c| + |V(F) \cap H| \leq (d-1)q^{m-1} + p_{m-1} = dq^{m-1} + p_{m-2}.$$

The assertion about $\max_F |V(F)|$ follows from the example of G_d given earlier. \square

Remark 2.3. Variants of Lemma 2.1 are seen elsewhere in the literature. For example, with notation as in Lemma 2.1, Homma [8, Prop. 2.2] has proved the following:

$$|X| \leq (a-1)q + 1 + \left\lfloor \frac{a-1}{p_{m-2}} \right\rfloor.$$

To see that this implies $|X| \leq aq + 1$, observe that $a \leq p_{m-1} = qp_{m-2} + 1$ and thus $\lfloor (a-1)/p_{m-2} \rfloor \leq q$. We can also deduce Lemma 2.1 from the Plotkin bound [15, Thm. 1.1.45] of coding theory. Indeed, we may assume without loss of generality that X is not contained in a hyperplane of \mathbb{P}^m (for otherwise, $|X| = a < aq + 1$). Now by [15, Thm. 1.1.6], $X \subset \mathbb{P}^m$ corresponds to a nondegenerate linear $[n, k, d]_q$ -code C with $n = |X|$, $k = m + 1$ and $d = n - a$. The Plotkin bound gives

$$d \leq \frac{nq^k(q-1)}{(q^k-1)q} = \frac{nq^m}{p_m}, \quad \text{which implies} \quad n \frac{p_{m-1}}{p_m} \leq a, \quad \text{that is, } n \leq \frac{ap_m}{p_{m-1}} \leq aq + 1.$$

In our proofs of Lemma 2.1 and Theorem 2.2, we have avoided any double counting argument. But if we were to permit it, then Lemma 2.1 can be proved more easily by counting the set $\{(P, H) \in X \times \widehat{\mathbb{P}}^m : P \in H\}$ in two ways using the first and the second projections, which yields $|X|p_{m-1} \leq ap_m$, and hence $|X| \leq aq + 1$.

3. TSFASMAN-BOGUSLAVSKY CONJECTURE FOR QUADRICS

In this section, we will consider the conjectural formula (5) for systems (4) where each F_i is homogeneous of degree 2. In other words, we consider intersections of quadrics. To begin with, observe that the maximum number of linearly independent homogeneous polynomials in $\mathbb{F}_q[x_0, \dots, x_m]$ of degree 2 is δ_m , where

$$(6) \quad \delta_j := \binom{j+2}{2} = 1 + 2 + \dots + (j+1) \quad \text{for any } j \in \mathbb{Z} \text{ with } j \geq -1.$$

Note that $0 = \delta_{-1} < 1 = \delta_0 < \delta_1 < \dots < \delta_m$. The following result is a restatement of [17, Thm. 3.4]. As usual, $\lfloor c \rfloor$ denotes the integer part of a real number c .

Theorem 3.1 (Zanella). *Let r be a positive integer $\leq \delta_m$, and let F_1, \dots, F_r be linearly independent homogeneous polynomials in $\mathbb{F}_q[x_0, \dots, x_m]$ of degree 2. If k is the unique integer such that $-1 \leq k < m$ and $\delta_m - \delta_{k+1} < r \leq \delta_m - \delta_k$, then*

$$(7) \quad |V(F_1, \dots, F_r)| \leq p_k + \lfloor q^{\epsilon-1} \rfloor, \quad \text{where} \quad \epsilon := \delta_m - \delta_k - r.$$

We shall deduce from Theorem 3.1 that if $d = 2$, then the TBC is true when $r \leq m + 1$ and false, in general, when $r > m + 1$ and $m > 2$.

Corollary 3.2. *Suppose $d = 2$ and $1 \leq r \leq m + 1$. Then (5) holds. In particular,*

$$\max_{F_1, \dots, F_r} |V(F_1, \dots, F_r)| = p_{m-1} + \lfloor q^{m-r} \rfloor = \begin{cases} p_{m-1} + q^{m-r} & \text{if } r = 1, \dots, m, \\ p_{m-1} & \text{if } r = m + 1. \end{cases}$$

where the maximum is over linearly independent homogeneous polynomials F_1, \dots, F_r of degree d in $\mathbb{F}_q[x_0, x_1, \dots, x_m]$.

Proof. For $1 \leq i \leq m + 1$, denote by e_i the $(m + 1)$ -tuple with 1 in i th place and 0 elsewhere. Lexicographically, the first $m + 1$ exponent vectors of polynomials of degree 2 in $m + 1$ variables are $e_1 + e_i$ for $i = 1, 2, \dots, m + 1$. Thus if $1 \leq r \leq m$, then the corresponding Tsfasman-Boguslavsky bound in (5) is given by

$$T_r = p_{m-2} + (p_{m-1} - p_{m-2}) + (p_{m-r} - p_{m-r-1}) = p_{m-1} + q^{m-r},$$

whereas if $r = m + 1$, then $T_r = p_{m-1} = p_{m-1} + \lfloor q^{m-r} \rfloor$. On the other hand, if $r \leq m + 1$, then the unique $k \in \mathbb{Z}$ satisfying $-1 \leq k < m$ and $\delta_m - \delta_{k+1} < r \leq \delta_m - \delta_k$ is clearly $k = m - 1$, and hence the corresponding Zanella bound in (7) is

$$Z_r = p_{m-1} + \lfloor q^{\delta_m - \delta_{m-1} - r - 1} \rfloor = p_{m-1} + \lfloor q^{m-r} \rfloor = T_r.$$

Thus Theorem 3.1 implies that $|V(F_1, \dots, F_r)| \leq T_r$ for $1 \leq r \leq m + 1$. It remains to show that the bound T_r is attained if $r \leq m + 1$. To this end, consider

$$Q_i = x_0 x_i \quad \text{for } i = 1, \dots, m \quad \text{and} \quad Q_{m+1} := x_0^2.$$

For any $r \leq m+1$, it is clear that Q_1, \dots, Q_r are linearly independent homogeneous polynomials of degree 2 in $\mathbb{F}_q[x_0, \dots, x_m]$. If $r \leq m$, then the common zeros in \mathbb{P}^m of Q_1, \dots, Q_r have homogeneous coordinates of the type $(a_0 : a_1 : \dots : a_m)$, where either (i) $a_0 = 0$ or (ii) $a_0 \neq 0$ and $a_1 = \dots = a_r = 0$. Hence

$$|V(Q_1, \dots, Q_r)| = p_{m-1} + q^{m-r} = T_r \quad \text{if } 1 \leq r \leq m.$$

In case $r = m+1$, the common zeros of Q_1, \dots, Q_r are only of the first type and so $|V(Q_1, \dots, Q_{m+1})| = p_{m-1} = T_{m+1}$. This completes the proof. \square

Remark 3.3. Roughly speaking, Corollary 3.2 shows that (5) holds for small values of r . It is not difficult to show that it also holds for a few large values of r . For instance, when $r = \delta_m$, the intersection of r linearly independent quadrics in \mathbb{P}^m is empty and thus has 0 elements, quite in accordance with (5). A little more generally, if $m \geq 2$, then one can see that the last 5 exponent vectors of monomials of degree 2 in $m+1$ variables are $(0, \dots, 0, 0, 2)$, $(0, \dots, 0, 1, 1)$, $(0, \dots, 0, 2, 0)$, $(0, \dots, 1, 0, 1)$, and $(0, \dots, 1, 1, 0)$. The corresponding Tsfasman-Boguslavsky bound in (5) is given, respectively, by

$$0, \quad 1, \quad 2, \quad q+1, \quad \text{and} \quad q+2.$$

One can check that these coincide with the corresponding Zanella bounds in (7). Moreover, it is easy to see that the bounds are attained by taking in the first case the set, say \mathcal{Q} of all monomials of degree 2 in $\mathbb{F}_q[x_0, x_1, \dots, x_m]$ and in the remaining four cases, taking the set obtained from \mathcal{Q} by successively dropping x_m^2 , x_{m-1}^2 , $x_{m-1}x_m$, and x_{m-2}^2 . Using these observations together with Corollary 3.2 we see as a special case that the TBC is true for quadrics in \mathbb{P}^2 , i.e., (5) holds if $d = 2$ and $m = 2$.

Corollary 3.4. *If $d = 2$ and $m > 2$, then (5) does not hold in general. In fact, (5) is false for at least $\binom{m-1}{2}$ values of positive integers r with $m+1 < r \leq \delta_m$.*

Proof. Let e_i ($1 \leq i \leq m+1$) be as in the proof of Corollary 3.2. Also let k be the unique integer such that $-1 \leq k < m$ and $\delta_m - \delta_{k+1} < r \leq \delta_m - \delta_k$. Write $r = \delta_m - \delta_{k+1} + i$ so that $1 \leq i \leq k+2$. Observe that the r th element, in descending lexicographic order, among the exponent vectors of monomials of degree 2 in $m+1$ variables, is precisely $e_{m-k} + e_{m-k+i-1}$. In particular its first nonzero coordinate is in the position $j := m - k$, and thus, with notation as in (2), the corresponding Tsfasman-Boguslavsky bound in (5) is given by

$$\begin{aligned} T_r &= p_{m-2j} + (p_{m-j} - p_{m-2j}) + (p_{m-(j+i-1)} - p_{m-j-(j+i-1)}) \\ &= p_{m-j} + p_{m-i-j+1} - p_{m-i-2j+1} \\ &= p_k + p_{k-i+1} - p_{2k-m-i+1}. \end{aligned}$$

On the other hand, the corresponding Zanella bound in (7) is given by

$$Z_r = p_k + \lfloor q^{\epsilon-1} \rfloor, \quad \text{where } \epsilon = \delta_m - \delta_k - (\delta_m - \delta_{k+1} + i) = \delta_{k+1} - \delta_k - i = k + 2 - i.$$

It follows that if $0 \leq k < m-1$ and $1 \leq i \leq k$, then

$$T_r - Z_r = p_{k-i} - p_{2k-m-i+1} \geq q^{k-i} > 0,$$

and so $Z_r < T_r$. Thus Theorem 3.1 implies that T_r can not be the maximum of $|V(F_1, \dots, F_r)|$ for arbitrary sets of r linear independent homogeneous polynomials of degree 2 in $\mathbb{F}_q[x_0, \dots, x_m]$. The number of such values of $r = \delta_m - \delta_{k+1} + i$ is

$$\sum_{k=0}^{m-2} \sum_{i=1}^k 1 = \sum_{k=0}^{m-2} k = \frac{(m-1)(m-2)}{2} = \binom{m-1}{2}.$$

Evidently this is positive if $m > 2$. This proves the corollary. \square

Example 3.5. The simplest case where the TBC is false seems to be that of intersections of 5 linearly independent quadrics in \mathbb{P}^3 . One can see in this case that Tsfasman-Boguslavsky bound is $T_5 = 2(1 + q)$, whereas the Zanella bound is $Z_5 = 1 + 2q$, which is strictly smaller.

Remark 3.6. The Tsfasman-Boguslavsky Conjecture stated in the Introduction (and abbreviated as TBC) is, in fact, a culmination of several conjectures that can be found in the paper of Boguslavsky [2], with at least one of the conjectures ascribed to Tsfasman. More precisely, the TBC is Corollary 5 of [2] whose hypothesis is that Conjecture 3 of [2] holds and whose “proof” uses Lemma 4 of [2]. For ease of reference, we state below Conjectures 1, 2 and 3 of [2]. To this end, let us first introduce some terminology. A projective variety X in \mathbb{P}^m over \mathbb{F}_q is said to be *linear* if its \mathbb{F}_q -rational points lie on the linear components of X . An m -tuple $(\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{Z}^m$ is said to be the *dim-type* of a projective variety X in \mathbb{P}^m if X has α_i irreducible components of codimension i for $i = 1, 2, \dots, m$. Given a finite family \mathcal{X} of projective varieties in \mathbb{P}^m , an element X of \mathcal{X} is said to be:

- (i) *maximal* in \mathcal{X} if $|X(\mathbb{F}_q)| = \max\{|Y(\mathbb{F}_q)| : Y \in \mathcal{X}\}$, and
- (ii) *dim-maximal* in \mathcal{X} if the dim-type of X is maximal (among the dim-types of all the elements of \mathcal{X}) with respect to the lexicographic order on \mathbb{Z}^m .

The conjectures in [2] concern the family, say \mathcal{X}_r , of projective varieties in \mathbb{P}^m defined by r linearly independent homogeneous polynomials in $\mathbb{F}_q[x_0, x_1, \dots, x_m]$ of degree d , and are as follows. Here $\Lambda(d, m)$ is as in the Introduction.

1. There exists a maximal family in \mathcal{X}_r which is linear.
2. If $(\nu_1, \dots, \nu_{m+1})$ is the r -th element of $\Lambda(d, m)$ in descending lexicographic order, then the dim-type of a dim-maximal element of \mathcal{X}_r is (ν_1, \dots, ν_m) .
3. There exists a maximal family in \mathcal{X}_r which is dim-maximal.

We remark that our positive result Corollary 3.2 and its proof (especially the examples therein) imply immediately that Conjectures 1 and 3 above hold in the affirmative when $d = 2$ and $r \leq m + 1$. Moreover, it is not difficult to also deduce Conjecture 2 in this case. On the other hand, the negative result in Corollary 3.4 does not necessarily imply that Conjectures 1, 2, and 3 above are false. It should also be remarked that the Tsfasman-Boguslavsky Conjecture stated in the Introduction has two aspects: (i) the expression on the right hand side of the equality in (5) is an upper bound for the number of common zeros of a system of r linearly independent homogeneous polynomials of degree d in $\mathbb{F}_q[x_0, x_1, \dots, x_m]$, and (ii) this upper bound is attained. Our negative result in Corollary 3.4 shows only that (ii) is false, but does not rule out the possibility that (i) holds, in general.

4. APPLICATIONS AND SUPPLEMENTS

In this first subsection below, we outline the relevance of TBC to coding theory, and in the second subsection, we provide a comparison with an older conjecture of Lachaud [6, Conj. 12.2] that is also stated, albeit with much too general hypothesis, by Boguslavsky [2, Conj. 4], and settled recently by Couvreur [3].

4.1. Projective Reed Muller Codes. Fix positive integers m, d and let $n := p_m$. Each point of $\mathbb{P}^m(\mathbb{F}_q)$ admits a unique representative in \mathbb{F}_q^{m+1} in which the first nonzero coordinate is 1. Let P_1, \dots, P_n be an ordered listing of such representatives in \mathbb{F}_q^{m+1} of points of $\mathbb{P}^m(\mathbb{F}_q)$. Denote by $\mathbb{F}_q[x_0, \dots, x_m]_d$ the space of homogeneous polynomials in $\mathbb{F}_q[x_0, \dots, x_m]$ of degree d together with the zero polynomial. Define

$$\text{PRM}_q(d, m) := \{(f(P_1), \dots, f(P_n)) : f \in \mathbb{F}_q[x_0, \dots, x_m]_d\}.$$

Evidently, this is a linear subspace of \mathbb{F}_q^n , and hence a q -ary (linear) code of length n . It is called the projective Reed-Muller code. This code is analogous to a more widely

studied class of codes called (affine or generalized) Reed-Muller code $\text{RM}_q(d, m)$. See, for example, [5, 7] for more on Reed-Muller codes and [1, Prop. 4] for a summary of several of its basic properties. The study of projective Reed-Muller codes was pioneered by Lachaud [9, 10] and Sørensen [14]. The relation with the TBC is through the notion of generalized Hamming weights, also known as higher weights, that goes back at least to Wei [16]. In general, for any q -ary linear code C of length n and dimension k , and any $D \subseteq C$, one defines

$$w_H(D) := |\{i \in \{1, \dots, n\} : c_i \neq 0 \text{ for some } c \in D\}|.$$

Now for $r = 1, \dots, k$, the r th higher weight of C is defined by

$$d_r(C) = \min\{w_H(D) : D \text{ is a subspace of } C \text{ with } \dim D = r\},$$

It may be remarked that $d_1(C)$ is what is called the minimum distance of C . The relationship of $\text{PRM}_q(d, m)$ with (5) is as follows: if $d < q$, then for $1 \leq r \leq \binom{m+d}{d}$,

$$(8) \quad d_r(\text{PRM}_q(d, m)) = p_m - \max_{F_1, \dots, F_r} |V(F_1, \dots, F_r)|,$$

where the maximum is over linearly independent F_1, \dots, F_r in $\mathbb{F}_q[x_0, x_1, \dots, x_m]_d$. To see (8) it suffices to use the relationship between linear codes and projective systems as described in [15, Thm. 1.1.14] and to note that when $d < q$, the code $\text{PRM}_q(d, m)$ corresponds to the projective system given by the \mathbb{F}_q -rational points of the Veronese variety corresponding to the Veronese embedding of \mathbb{P}^m of degree d . Thus it is clear that the TBC admits an equivalent statement in terms of an explicit formula for the higher weights of projective Reed-Muller codes. In particular, (8) and Corollary 3.2 imply that

$$(9) \quad d_r(\text{PRM}_q(2, m)) = q^m - \lfloor q^{m-r} \rfloor \quad \text{for } r = 1, \dots, m+1$$

whereas Remark 3.3 shows that

$$d_{\delta_m-r}(\text{PRM}_q(2, m)) = \begin{cases} p_m - r & \text{if } r = 0, 1, 2, \\ p_m - (q + r - 2) & \text{if } r = 3, 4. \end{cases}$$

It may be noted that (9) can be viewed as a generalization of the last theorem in [9]. We end this subsection by remarking that an affine analogue of the Tsfasman-Boguslavsky Conjecture is true, in general for $1 < d < q$, thanks to the complete determination of all higher weights of the Reed-Muller code $\text{RM}_q(d, m)$ by Heijnen and Pelikaan [7, Thm. 5.10].

4.2. Comparison with a Theorem of Couvreur. In a recent preprint [3], Couvreur has proved the following result, answering as a special case a conjecture that goes back to Lachaud and stated in [6, Conj. 12.2] (see also [11, Conj. 5.3]).

Theorem 4.1 (Couvreur). *Let X be a nondegenerate projective variety in \mathbb{P}^m defined over \mathbb{F}_q . Suppose the irreducible components of X have dimensions n_1, \dots, n_t and degrees $\delta_1, \dots, \delta_t$, respectively. If $n_i < m$ for all $i = 1, \dots, t$, then*

$$(10) \quad |X(\mathbb{F}_q)| \leq p_{2n-m} + \sum_{i=1}^t \delta_i (p_{n_i} - p_{2n_i-m}) \quad \text{where } n := \max\{n_1, \dots, n_t\}.$$

In particular, if X is equidimensional of dimension n and degree δ , then

$$(11) \quad |X(\mathbb{F}_q)| \leq \delta p_n - (\delta - 1)p_{2n-m} = \delta(p_n - p_{2n-m}) + p_{2n-m}.$$

Moreover, the upper bound is optimal for equidimensional varieties.

The original conjecture by Lachaud assumed X be a complete intersection (and hence equidimensional) of degree $\delta \leq q + 1$ and had an additional hypothesis that $2n \geq m$, lest the bound in (11) reduces to a known inequality (cf. [2, Thm. 3], [6, Prop. 12.1]). Just like the TBC, the conjecture by Lachaud reduces to Serre's

inequality (3) when $\text{codim } X = m - n = 1$. But for $\text{codim } X > 1$, the relation between two conjectures above may not appear sufficiently clear and it may be worthwhile to try to make it clearer. First, it should be noted that the hypothesis of TBC is amenable to an easy verification—one just have to check that the defining equations have the same degree and are linearly independent. On the other hand, determining the dimensions and degrees of irreducible components from a given set of equations defining the variety can be quite difficult. In fact, even when the variety is known to be irreducible, determining the degree may not be easy, unless of course it is a hypersurface. One basic case where the hypotheses of the TBC and Theorem 4.1 coincide and are easily checked is when $X \subseteq \mathbb{P}^m$ is defined by the vanishing of r linearly independent homogeneous polynomials in $m + 1$ variables, each of the same degree d , and $n = \dim X = m - r$ so that X is a complete intersection. In this case X is equidimensional and has degree $\delta = d^r$. Assume, for simplicity, that $n \geq 0$, i.e., $r \leq m$ and that $d > 1$ and $\delta \leq q + 1$. Then $(d - 1, 0, \dots, 0, 1, 0, \dots, 0)$, with 1 is in the r -th place, is the r th element of $\Lambda(d, m)$. Consequently, the Tsfasman-Boguslavsky bound, say $T_r(d)$, on $|X(\mathbb{F}_q)|$ is equal to $p_{m-2} + (d - 1)(p_{m-1} - p_{m-2}) + (p_{m-r} - p_{m-r-1}) = (d - 1)q^{m-1} + q^{m-r} + p_{m-2}$.

Also since $d - 1 \geq 1$, putting $p_{m-2} = (q^{m-1} - 1)/(q - 1)$, we see that

$$(12) \quad T_r(d) \geq \frac{q^m + q^{m-r+1} - q^{m-r} - 1}{q - 1}.$$

On the other hand, the Couvreur bound in (11), say $C_r(d)$, in this situation is

$$d^r(p_{m-r} - p_{m-2r}) + p_{m-2r} = (d^r - 1)(p_{m-r} - p_{m-2r}) + p_{m-r}.$$

Since $d^r \leq q + 1$, i.e., $d^r - 1 \leq q$, we easily see that

$$C_r(d) \leq \frac{q^{m-2r+2}(q^r - 1) + (q^{m-r+1} - 1)}{q - 1} = \frac{q^{m-r+2} + q^{m-r+1} - q^{m-2r+2} - 1}{q - 1}.$$

Comparing the right hand side of the above equation with (12), we see that $C_r(d) \leq T_r(d)$ if $r \geq 2$. It follows that the Couvreur bound is sharper, especially when $d > 2$ and $r > 2$. Of course, this, by itself, does not contradict the TBC since the projective varieties where the Tsfasman-Boguslavsky bounds are attained are seldom equidimensional (that is to say, having all its irreducible components of the same dimension), let alone complete intersections. Indeed, in the commonly applicable situation considered above, projective varieties attaining the Tsfasman-Boguslavsky bound is expected to have $d - 1$ common components of codimension 1 and one of codimension r . As Couvreur [3, §5.2] has remarked, his bound in the non-equidimensional case might not be optimal. This is, in fact, true, and to see this, one can consider $d = 2$, $r \leq m$, and the example of quadrics Q_1, \dots, Q_r in the proof of Corollary 3.2. As we have seen, the projective variety, say X , cut out by these quadrics has $p_{m-1} + q^{m-r}$ points. Also it is clear that X has two irreducible components, the hyperplane $x_0 = 0$ and the linear subspace $x_1 = \dots = x_r = 0$. Both the components are linear and are complete intersections. Thus in the notation of Theorem 4.1, we have $t = 2$ and $n_1 = m - 1$, $n_2 = m - r$, while $\delta_1 = 1 = \delta_2$. It follows that the upper bound of Couvreur in (10) in this case is

$$p_{2(m-1)-m} + (p_{m-1} - p_{2(m-1)-m}) + (p_{m-r} - p_{2(m-r)-m}),$$

or in other words,

$$p_{m-1} + q^{m-r} + q^{m-r-1} + \dots + q^{m-2r+1}.$$

So if $r \geq 2$, then the Tsfasman-Boguslavsky bound is sharper than the Couvreur bound in this case. It is thus seen that the two bounds compliment each other and neither implies the other, in general.

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